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A Note on the Finite Sample Properties of the CLS Method of TAR Models

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Abstract

In this paper we focus on the finite sample properties of the conditional least squares (CLS) method of threshold autoregressive (TAR) parameters under the following conditions: (a) non-Gaussian model innovations; (b) two types of asymmetry (i.e. deepness and steepness) captured by TAR models. It is clearly demonstrated that the finite sample properties of the CLS method of TAR parameters significantly differ depending on the type of asymmetry. The behavior of steepness-based models is very good compared to that obtained from deepness-based models. Therefore, extreme caution must be exercised to preliminary modelling steps, such as testing the type of asymmetry before estimating TAR models in practice. A mistake in this phase of modelling can, in turn, give rise to very problematic results.

1 Introduction

There is overwhelming empirical evidence in the literature that many economic variables do exhibit some form of asymmetry and/or non-linearity.[1] As a result, many interesting

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[1]See, for example, Neftci (1984), Sichel (1989, 1993), Verbrugge (1997), Razzak (2001), Tiao and Tsay (1994) and Potter (1995), Peel and Speight (1998a,b), among others.

non-linear time series models have been proposed in the literature.[2] Threshold autoregressive (TAR) models, proposed by Tong and Lim (1980), are one particular class of regime-switching models, which has become very popular in time series econometrics. Hansen (2011) provides a survey of 75 TAR model applications in macroeconomics and finance. Although the limiting properties of the estimated TAR parameters are nowadays well established, Kapetanios (2000) showed that the conditional least squares (CLS) method performs quite poorly in finite samples, especially for the threshold parameter. Norman (2008) demonstrated that a bias of the threshold parameter is related with allocation of observations into individual regimes.[3] In addition, Coakley et al. (2003) discussed computational aspects of the CLS method. The main task of this short note is to extend the previous results and assess the finite sample properties of the CLS method of TAR parameters under the following conditions: (a) non-Gaussian model innovations; (b) two types of asymmetry (i.e. deepness and steepness) captured by TAR models.

The paper is organized as follows. A brief description of the CLS method is given in Section 2. Monte Carlo setup and results are presented in Sections 3 and 4.

2 Threshold autoregressive models

Without loss of generality, we consider a 2-regime TAR model with a symmetric lag structure, denoted as TAR$(2; p, p)$. The model is formally written as follows

$$X_t = \boldsymbol{\phi}_1' \mathbf{X}_{t-1} I(X_{t-d} \leq c) + \boldsymbol{\phi}_2' \mathbf{X}_{t-1} I(X_{t-d} > c) + \epsilon_t, \tag{1}$$

where $\{\epsilon_t : t \in \mathbb{Z}\}$ is a sequence of IID$(0,\sigma^2)$ model innovations; d is the delay parameter; $\mathbf{X}_{t-1} = (1, X_{t-1}, \ldots, X_{t-p})'$ denotes a $(p+1 \times 1)$ vector of predetermined variables; $\boldsymbol{\phi}_i = (\phi_{i0}, \phi_{i1}, \ldots, \phi_{ip})'$ denotes a $(p+1 \times 1)$ vector of unknown parameters for the regime $i \in \{1, 2\}$. Chen and Tsay (1993) derive a stationarity condition for higher-order TAR models, which is similar to that for linear AR(p) models. The derivation of basic moments of TARMA models is discussed in Amendola et al. (2006).

A convenient way to estimate a TAR model defined in (1) is to apply a sequential conditional least squares (CLS) method, which is based on the fact that for pre-determined lag order p, the delay parameter d, and the fixed threshold parameter c, the model is

[2]See Hamilton (1989) for Markov switching autoregressive (MSAR) models; Tong and Lim (1980) for threshold autoregressive (TAR) models; Teräsvirta (1994) for smooth transition autoregressive (STAR) models, Wong and Li (2000) for mixture autoregressive (MAR) models.

[3]To be precise, a bias is positively correlated with the average percentage of observations in the upper regime.

linear in remaining parameters. The estimate of $\boldsymbol{\phi} = (\boldsymbol{\phi}_1', \boldsymbol{\phi}_2')'$ can be obtained by the CLS as follows

$$\hat{\boldsymbol{\phi}}(c) = \left(\sum_{t=1}^{T} \mathbf{X}_t(c)\mathbf{X}_t(c)'\right)^{-1} \left(\sum_{t=1}^{T} \mathbf{X}_t(c)X_t\right),$$

where $\mathbf{X}_t(c) = (\mathbf{X}_t' I(X_{t-d} \leq c), \mathbf{X}_t' I(X_{t-d} > c))'$ and the notation $\hat{\boldsymbol{\phi}}(c)$ indicates that the estimate is conditional on the pre-specified threshold value c. The corresponding (conditional) residual variance is defined as

$$\hat{\sigma}^2(c) = \frac{1}{T}\sum_{t=1}^{T}(X_t - \hat{\boldsymbol{\phi}}(c)'\mathbf{X}_t(c))^2.$$

The CLS estimate of the threshold parameter c is obtained by minimizing the (conditional) residual variance $\hat{\sigma}^2(c)$ using a grid search, that is

$$\hat{c} = \operatorname*{argmin}_{c \in C} \hat{\sigma}^2(c),$$

where $C = [\xi_{k_1}, \ldots, \xi_{k_2}]$ is a compact set, ξ_k is a particular sample quantile set in such a way to ensure the sufficient number of observations in each regime.

Under relatively mild conditions[4], it can be shown that the limiting distribution of the AR parameters of a TAR model is normal, yet the limiting distribution of the threshold parameter c depends on whether a TAR model is continuous or not. In the continuous case, the limiting distribution of the threshold parameter is normal as well, whereas in the discontinuous case, the limiting distribution is a complicated compounded Poisson distribution, see Chan (1993). Although the limiting properties of estimated parameters of a TAR model by the CLS method are known, their finite sample properties are problematic. Kapetanios (2000) shows that the CLS method performs quite poorly in finite samples, especially in the case of the threshold parameter c.

3 Monte Carlo setup

Although there is very likely no uniformly correct way to specify the setup of Monte Carlo experiments, the results based on simple first-order TAR models with Gaussian innovations might be of the limited applicability for empirical research. For this reason, the Monte Carlo setup in this paper is based on empirically estimated TAR models capturing two different types of asymmetry/non-linearity (i.e. deepness and steepness)

[4]See Condition 1 – 4 in Chan (1993, p. 522-523).

usually observed in economic time series.[5] It is assumed that a given economic variable y_t can be decomposed into a trend component τ_t and a cyclical component x_t such that: $y_t = \tau_t + x_t$.[6] TAR models based on cyclical component x_t capture deepness, while TAR models based on first differences Δy_t capture steepness. The deepness-based TAR models are denoted as "D" models, whereas the steepness-based models are denoted as "S" models, see Table 1.[7] Figure 1 depicts the cyclical component x_t and the growth rates Δy_t of the US real GDP series.

The finite sample properties of the CLS estimator are examined based on various distributions of innovations. In particular, apart from a standard normal distribution, which serves as a benchmark for comparison, we consider model innovations $\chi^2(5)$ and $t(5)$ distributions.[8] All model innovations are standardized to have zero mean and unit variance.

Table 1: Empirical TAR models used in Monte Carlo experiments

---

D1:

$$x_t = \begin{cases} -0.4 + 0.68x_{t-1} + 0.17x_{t-2} + \epsilon_t & \text{for} \quad x_{t-1} \leq -1.8, \\ 1.40x_{t-1} - 0.45x_{t-2} + \epsilon_t & \text{for} \quad x_{t-1} > -1.8, \end{cases}$$

from Peel and Speight (1998b, p. 329) fitted to real German GDP.

D2:

$$x_t = \begin{cases} -1.37 + 1.30x_{t-1} - 0.61x_{t-2} + \epsilon_t & \text{for} \quad x_{t-1} \leq -2.2, \\ 0.10 + 1.10x_{t-1} - 0.18x_{t-2} + \epsilon_t & \text{for} \quad x_{t-1} > -2.2, \end{cases}$$

from Peel and Speight (1998b, p. 329) fitted to real US GNP.

S1:

$$\Delta y_t = \begin{cases} -0.59 - 1.20\Delta y_{t-1} + 0.52\Delta y_{t-2} + \epsilon_t & \text{for} \quad \Delta y_{t-1} \leq -0.10, \\ 0.57 + 0.22\Delta y_{t-1} + \epsilon_t & \text{for} \quad \Delta y_{t-1} > -0.10, \end{cases}$$

from Peel and Speight (1998b, p. 330) fitted to real German GDP.

[5]For example, Verbrugge (1997) examines 11 US economic time series. He concludes that 8 time series exhibit statistically significant deepness, 6 time series steepness, and 4 series both.

[6]The cyclical component x_t is often extracted by some band-pass filter.

[7]Note that higher order TAR models were transformed into the form of TAR(2;2,2) for the purpose of Monte Carlo experiments.

[8]Note that: (i) if $\epsilon \sim t(5)$ then the coefficient of skewness is 0.0 and the coefficient of kurtosis is 9.0; (i) if $\epsilon \sim \chi^2(5)$ then the coefficient of skewness is 1.3 and the coefficient of kurtosis is 5.4. We consider only $\chi^2(5)$ with positive skewness to ensure the sufficient number of observations in the lower (recession) regime.

S2:

$$\Delta y_t = \begin{cases} -0.39 + 0.44\Delta y_{t-1} - 0.79\Delta y_{t-2} + \epsilon_t & \text{for} \quad \Delta y_{t-1} \le 0.0, \\ 0.38 + 0.31\Delta y_{t-1} + 0.20\Delta y_{t-2} + \epsilon_t & \text{for} \quad \Delta y_{t-1} > 0.0, \end{cases}$$

from Tiao and Tsay (1994, p. 113) fitted to real US GDP.

Note: x_t denotes the cyclical component used for modelling deepness, whereas Δy_t represents the growth rates used for modelling steepness.

Originally, T+100 observations are simulated in each experiment, but the first 100 of them are discarded to eliminate the effect of initial observations. The number of repetitions in all experiments is set to $R = 2,000$, and the number of observations is set to $T \in \{100, 300, 500\}$. We follow a conventional assumption that the lag order p and the delay parameter d of TAR models are both known, whereas the threshold parameter is estimated via a 100-point grid search with the set $C = [\xi_{0.1}, \ldots, \xi_{0.9}]$.[9] Following arguments in Coakley et al. (2003), the CLS method is based on the QR factorization.

4 Monte Carlo results

The following quantities about the estimated TAR parameters are considered:

$$bias = \frac{1}{R}\sum_{r=1}^{R}(\hat{\xi}_r - \xi),$$

'bias" stands for the average bias of the estimated parameter, say, $\hat{\xi}$ calculated over all replications;

$$mse = \frac{1}{R}\sum_{r=1}^{R}(\hat{\theta}_r - \theta)^2,$$

"mse" denotes a mean square error of the estimated parameter calculated over all replications;

$$jb = sk^2/6 + (kt - 3)^2/24,$$

[9]As the parameter space of the delay parameter d is discrete, the CLS estimate is super-consistent, and thus d can be consider as known once the switching variable is identified. For this reason, a grid search over the delay parameter d is not considered in this paper.

"jb" denotes the Jarque-Bera test statistic, and sk and kt are sample coefficients of skewness and kurtosis calculated from estimated parameter over all replications;

$$\pi = \frac{1}{R}\sum_{r=1}^{R}\pi_r,$$

denotes the average proportion of observations lying in the lower (i.e. recession) regime over all replications, where $\pi_r = T^{-1}\sum_{t=1}^{T} I(z_t < c)$;

$$q = \frac{1}{R}\sum_{r=1}^{R} q_r,$$

q stands for the average percentage regime mismatch calculated aver all replications, where $q_r = T^{-1}\sum_{t=1}^{T} I(z_t < c) - I(z_t < \hat{c}))^2$. The p-values of the Jarque-Bera test are presented in tables below.

4.1 Bias

The Monte Carlo results are presented in Table 2. The results suggest the following. The bias of the threshold parameter c and regime constants ϕ_{i0} of deepness-based TAR models (D1,D2) are significantly larger compared to those of steepness-based TAR models (S1,S2). For example, the bias of the threshold parameter of deepness-based models are 0.94 and 1.21 compared to 0.13 and -0.07 for steepness-based models in the sample $T = 100$ and using Gaussian innovations. Moreover, it is worth noting that the bias of key TAR parameters is not directly related to the number of observations allocated in individual regimes, as claimed by Norman (2008), but rather to the persistence of the switching (indicator) function. For example, D2 and S1 models have a very similar allocation of observations in regimes[10], but the bias of the deepness-based model (D2) is larger by the factor 10 compared to the bias of the steepness-based model (S1). The reason for that lies not in the proportion of observations allocated in individual regimes, but rather in the persistence of the switching (i.e. indicator) function, see Figures 2-3, where examples of switching (indicator) functions and their sample autocorrelations of all four DGPs are depicted. Our findings reveal that the higher the persistence of the switching function, the higher the probability of a regime mismatch, and subsequently the higher the bias of the threshold parameter c. This fact makes the deepness-based models very problematic in small samples.

***** Insert Table 2 around here *****

***** Insert Figure 1 and 2 around here *****

[10] A probability of the process being in the lower (recession) regime is approximately 25 %.

It is also interesting to note that steepness-based TAR models do not exhibit any significant sensitivity on the specification of model innovations regardless the sample size and the setup of the CLS method, whereas deepness-based TAR models are much more sensitive. However, it is interesting to note that the CLS method produces smaller or equal bias of the TAR parameters in approximately 80 % of the cases, provided that innovations are drawn from a non-Gaussian but symmetric distribution (i.e. $t(5)$), and in approximately 60 % of the cases, provided that innovations are non-Gaussian but asymmetric (i.e. $\chi^2(5)$).

4.2 Mean square error

The mean square errors (MSE) of the estimated TAR parameters are presented in Table 3. Since the MSE of the estimated TAR parameters is significantly affected by the bias of a given TAR parameter, the bias and MSE results are very similar. Again, we can observe a large difference between MSE of deepness-based and steepness-based TAR models. For example, the MSE of the threshold parameter c od steepness-based models is less than 0.25 in the smaple $T = 100$, but more than 4.9 for deepness-based models. Similar results are observed for other TAR parameters, especially regime constants in monor regimes, see Table 3. Moreover, very interesting finding is that the MSE of the deepness-based models is very sensitive on the specification of model innovations. For example, the MSE of the regime constant ϕ_{10} of the deepness-based TAR model D1 is 0.62 for Gaussian innovation, 1.67 for t innovations, even 2.59 for χ^2 innovations in the sample $T = 500$, see top-right panel of Table 3. Rather surprisingly, no similar sensitivity is observed for steepness-based TAR models regardless of the sample size.

***** Insert Table 3 around here *****

4.3 Normality

The Monte Carlo results are presented in Table 4. Using the standard Jarque-Bera test, normality of the estimated TAR parameters is clearly rejected in 100 % of the cases in the sample $T = 100$ at 5 % significance level, and in 70 % of the cases for the sample $T = 300$ for all DGP configurations (D1, D2, S1, S2) and model innovations ($N(0,1)$, $t(5)$, $\chi^2(5)$). However, large differences can be observed again between deepness-based and steepness-based TAR models. For example, for the former TAR model, normality is rejected in 95 % of the cases, whereas only in 40 % in the latter one in the sample $T = 500$ and at 5 % significance level. This finding clearly documents how much different the properties of the estimated TAR parameters from deepness-based and steepness-based models can be. Moreover, it can be concluded that rejecting normality is affected by the allocation of observations into individual regimes, see results for both steepness-based models (S1 and S2) in Table 4. Loosely speaking, normality is very likely to be rejected for TAR parameters related to a minor regime.

***** Insert Table 4 around here *****

5 Conclusion

It is shown that the finite sample properties of the CLS method of TAR parameters significantly differ depending on the type of asymmetry. Our results clearly indicate that steepness-based TAR models produce relatively small bias and MSE, which are significantly reduced with the increasing sample size. Both quantities are insensitive to a distribution of TAR innovations. Moreover, it is shown that the limiting distribution of steepness-based TAR parameters is a relatively good approximation in the smaples $T \geq 500$. Unfortunately, no one from the above properties holds for the deepness-based TAR parameters. Therefore, extreme caution must be exercised to preliminary modelling steps, such as testing a particular type of asymmetry. A mistake in this phase of modelling can, in turn, give rise to completely different, and rather problematic, results.

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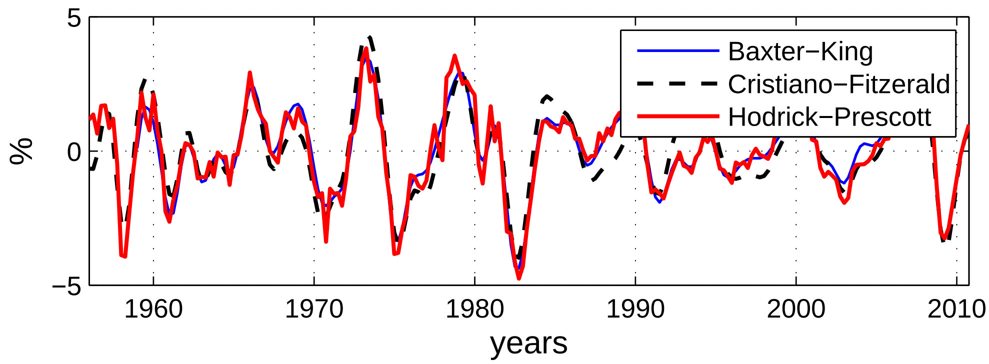
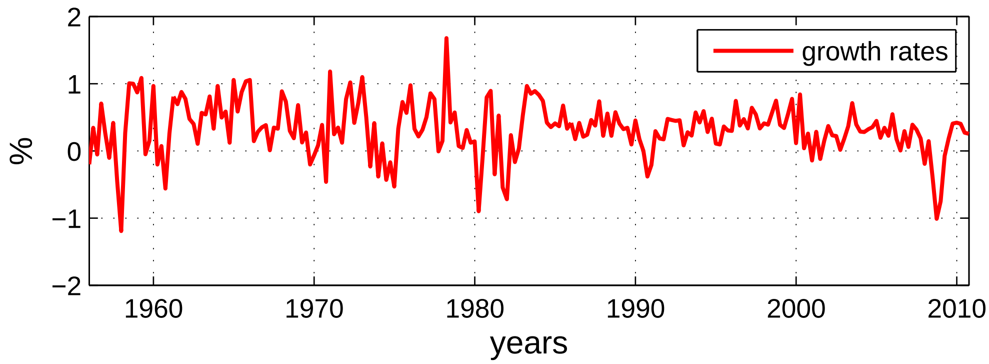
Figure 1: Example of the real US GDP series



(a) cyclical component x_t



(b) growth rates Δy_t

Note: The cyclical component x_t is extracted/calculated using the following three filters: the (symmetric) Baxter-King filter, the (asymmetric) Christiano-Fitzerald filter, and Hodrick-Prescott filter.

Figure 2: Examples of switching (indicator) functions of TAR models: $T = 100$

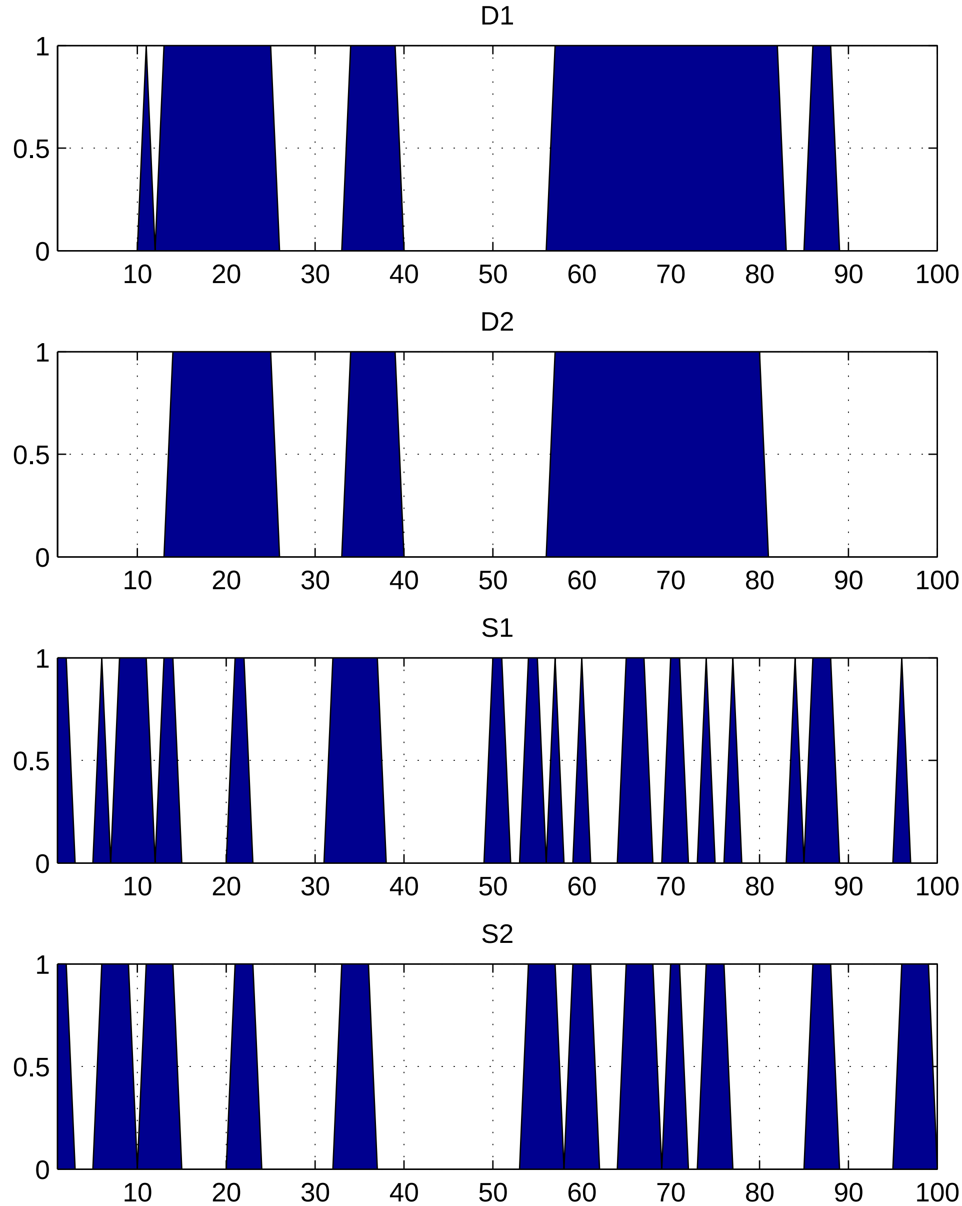


Figure 3: Examples of sample autocorrelations of switching (indicator) functions of TAR models: $T = 100$

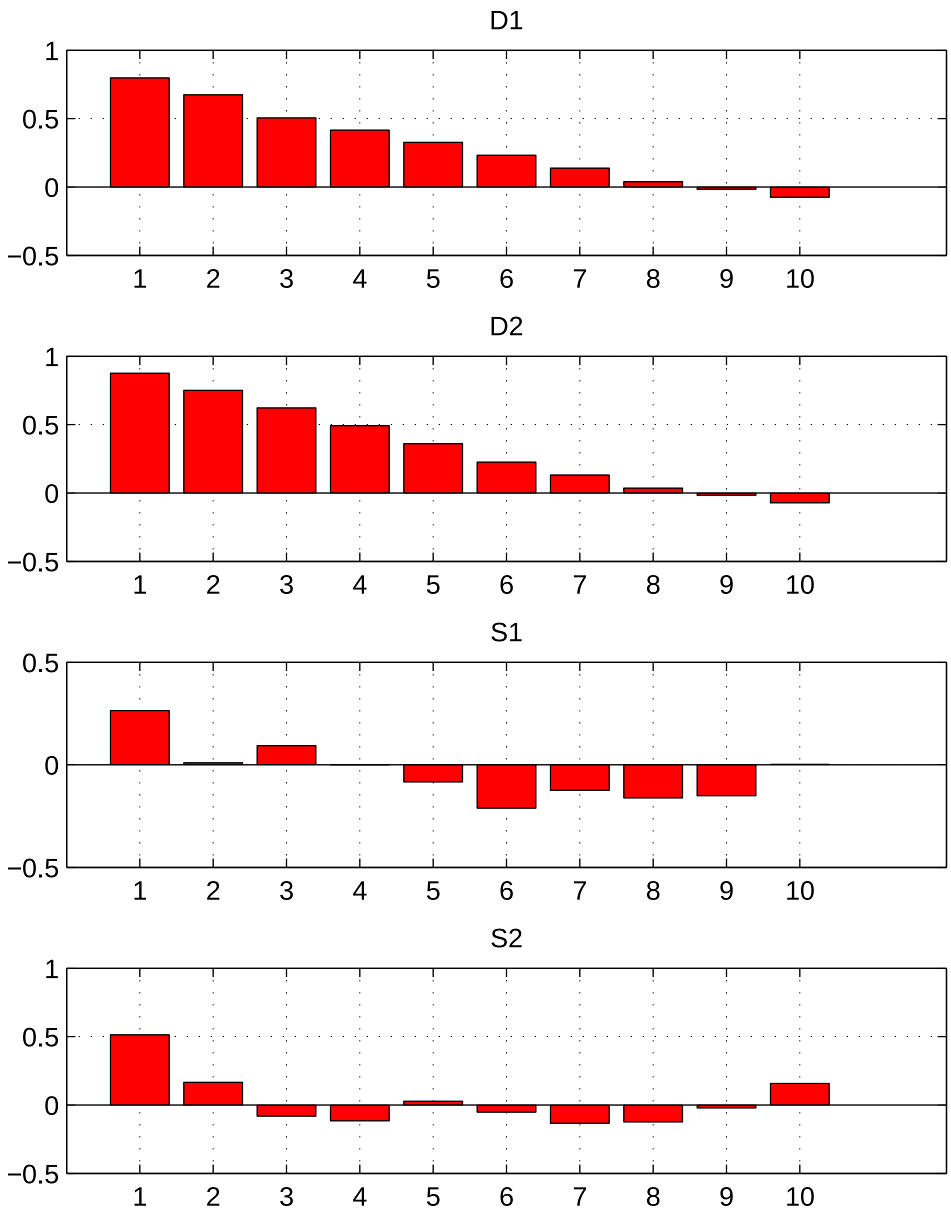


Table 2: Finite sample properties of the CLS method: bias

		T=100			**T=300**			**T=500**		
model	params.	$N(0,1)$	$t(5)$	χ^2	$N(0,1)$	$t(5)$	χ^2	$N(0,1)$	$t(5)$	χ^2
D1	ϕ_{10}	-0.63	-0.69	-0.82	-0.45	-0.42	-0.50	-0.23	-0.22	-0.32
	ϕ_{11}	-0.04	-0.06	-0.13	-0.11	-0.09	-0.12	-0.06	-0.05	-0.08
	ϕ_{12}	-0.27	-0.27	-0.26	-0.04	-0.04	-0.05	-0.01	-0.02	-0.01
	ϕ_{20}	0.21	0.17	0.19	0.03	0.02	0.04	0.01	0.01	0.01
	ϕ_{21}	-0.05	-0.04	-0.05	-0.01	-0.01	-0.01	0.00	-0.01	-0.01
	ϕ_{22}	-0.02	-0.01	-0.01	0.00	0.00	0.00	0.00	0.00	0.00
	c	0.94	0.93	0.76	-0.10	-0.14	-0.17	-0.10	-0.13	-0.19
	π	0.19	0.17	0.18	0.19	0.19	0.18	0.19	0.18	0.19
	q	0.17	0.18	0.15	0.04	0.03	0.04	0.02	0.02	0.02
D2	ϕ_{10}	0.26	0.28	0.37	-0.06	-0.06	-0.07	-0.04	-0.04	-0.07
	ϕ_{11}	-0.09	-0.08	-0.11	-0.01	0.00	-0.01	0.00	0.00	-0.01
	ϕ_{12}	0.08	0.07	0.08	-0.01	-0.01	-0.02	-0.01	-0.01	-0.01
	ϕ_{20}	0.21	0.17	0.33	0.03	0.02	0.04	0.01	0.01	0.01
	ϕ_{21}	-0.06	-0.05	-0.07	-0.01	-0.01	-0.01	-0.01	-0.01	0.00
	ϕ_{22}	-0.04	-0.03	-0.03	-0.01	-0.01	0.00	0.00	0.00	0.00
	c	1.21	1.17	1.33	0.13	0.01	0.07	0.00	-0.01	0.00
	π	0.29	0.26	0.22	0.28	0.27	0.23	0.28	0.28	0.24
	q	0.18	0.18	0.19	0.03	0.02	0.02	0.01	0.01	0.01
S1	ϕ_{10}	0.06	0.06	0.03	0.00	0.00	0.00	0.01	0.00	0.01
	ϕ_{11}	0.04	0.04	0.03	-0.01	-0.01	0.00	0.00	0.00	0.01
	ϕ_{12}	-0.03	-0.03	0.01	0.00	-0.01	-0.01	-0.01	-0.01	-0.01
	ϕ_{20}	0.11	0.09	0.10	0.02	0.01	0.01	0.00	0.00	0.00
	ϕ_{21}	-0.05	-0.04	-0.03	-0.01	-0.01	-0.01	0.00	0.00	0.00
	ϕ_{22}	-0.03	-0.03	-0.04	-0.01	-0.01	-0.01	0.00	0.00	0.00
	c	0.13	0.09	0.08	0.00	-0.01	0.00	0.00	0.00	0.00
	π	0.24	0.22	0.31	0.25	0.22	0.32	0.25	0.22	0.31
	q	0.08	0.07	0.07	0.01	0.01	0.01	0.01	0.01	0.01
S2	ϕ_{10}	-0.04	-0.07	-0.03	-0.01	-0.01	0.00	0.00	0.01	0.01
	ϕ_{11}	-0.02	-0.03	0.00	-0.01	-0.01	0.00	0.00	0.00	0.01
	ϕ_{12}	-0.04	-0.05	-0.07	-0.01	-0.01	-0.02	0.00	0.00	0.00
	ϕ_{20}	0.02	0.03	0.03	0.00	0.00	0.00	0.00	0.00	0.00
	ϕ_{21}	-0.02	-0.02	-0.01	0.00	0.00	0.00	0.00	0.00	0.00
	ϕ_{22}	-0.02	-0.03	-0.03	-0.01	-0.01	-0.01	-0.01	-0.01	0.00
	c	-0.07	-0.07	-0.05	-0.01	-0.01	-0.01	0.00	0.00	0.00
	π	0.32	0.29	0.31	0.31	0.28	0.31	0.31	0.28	0.31
	q	0.02	0.02	0.02	0.01	0.01	0.01	0.00	0.01	0.01

* "bias" stands for the average bias, "mse" denotes a mean squared error.

Table 3: Finite sample properties of the CLS method: mse

		T=100			**T=300**			**T=500**		
model	params.	$N(0,1)$	$t(5)$	χ^2	$N(0,1)$	$t(5)$	χ^2	$N(0,1)$	$t(5)$	χ^2
D1	ϕ_{10}	2.60	3.25	5.31	1.68	1.90	4.56	0.62	1.67	2.59
	ϕ_{11}	0.27	0.31	0.67	0.14	0.11	0.38	0.04	0.09	0.21
	ϕ_{12}	0.21	0.21	0.40	0.06	0.05	0.16	0.02	0.02	0.09
	ϕ_{20}	0.77	0.98	0.54	0.10	0.13	0.19	0.00	0.07	0.01
	ϕ_{21}	0.05	0.07	0.03	0.01	0.01	0.01	0.00	0.00	0.00
	ϕ_{22}	0.02	0.03	0.01	0.00	0.00	0.00	0.00	0.00	0.00
	c	4.95	4.67	4.13	0.89	0.84	1.24	0.28	0.25	0.43
D2	ϕ_{10}	1.71	2.54	3.08	0.39	0.73	0.81	0.11	0.09	0.55
	ϕ_{11}	0.11	0.16	0.37	0.02	0.03	0.06	0.01	0.01	0.03
	ϕ_{12}	0.08	0.09	0.17	0.02	0.02	0.03	0.01	0.01	0.02
	ϕ_{20}	1.27	1.54	1.34	0.24	0.22	0.14	0.01	0.00	0.02
	ϕ_{21}	0.09	0.11	0.05	0.01	0.01	0.01	0.00	0.00	0.00
	ϕ_{22}	0.04	0.04	0.02	0.01	0.01	0.00	0.00	0.00	0.00
	c	7.18	6.41	6.83	1.08	0.60	0.84	0.11	0.08	0.20
S1	ϕ_{10}	0.38	0.35	0.36	0.05	0.04	0.03	0.02	0.13	0.02
	ϕ_{11}	0.35	0.29	0.65	0.05	0.04	0.07	0.03	0.06	0.04
	ϕ_{12}	0.08	0.12	0.23	0.02	0.02	0.03	0.01	0.01	0.02
	ϕ_{20}	0.71	0.22	0.30	0.02	0.01	0.03	0.01	0.01	0.01
	ϕ_{21}	0.14	0.05	0.05	0.01	0.01	0.01	0.00	0.00	0.00
	ϕ_{22}	0.02	0.02	0.02	0.00	0.00	0.00	0.00	0.00	0.00
	c	0.22	0.15	0.15	0.01	0.01	0.01	0.00	0.00	0.00
S2	ϕ_{10}	0.22	0.35	0.43	0.04	0.04	0.04	0.02	0.02	0.02
	ϕ_{11}	0.10	0.13	0.23	0.02	0.02	0.04	0.01	0.01	0.02
	ϕ_{12}	0.05	0.07	0.12	0.01	0.01	0.02	0.01	0.01	0.01
	ϕ_{20}	0.06	0.05	0.05	0.01	0.01	0.01	0.01	0.01	0.01
	ϕ_{21}	0.03	0.02	0.02	0.01	0.01	0.00	0.00	0.00	0.00
	ϕ_{22}	0.01	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00
	c	0.04	0.05	0.05	0.00	0.00	0.00	0.00	0.00	0.00

* "bias" stands for the average bias, "mse" denotes a mean squared error.

Table 4: Finite sample properties of the CLS method: normality

		T=100			**T=300**			**T=500**		
model	params.	$N(0,1)$	$t(5)$	χ^2	$N(0,1)$	$t(5)$	χ^2	$N(0,1)$	$t(5)$	χ^2
D1	ϕ_{10}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	ϕ_{11}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	ϕ_{12}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	ϕ_{20}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	ϕ_{21}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	ϕ_{22}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	c	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
D2	ϕ_{10}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	ϕ_{11}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	ϕ_{12}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	ϕ_{20}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	ϕ_{21}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.12	0.00
	ϕ_{22}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.97	0.00
	c	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S1	ϕ_{10}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	ϕ_{11}	0.00	0.00	0.00	0.00	0.00	0.00	0.15	0.00	0.00
	ϕ_{12}	0.00	0.00	0.00	0.00	0.00	0.00	0.72	0.00	0.00
	ϕ_{20}	0.00	0.00	0.00	0.00	0.02	0.00	0.07	0.69	0.05
	ϕ_{21}	0.00	0.00	0.00	0.03	0.21	0.00	0.25	0.43	0.47
	ϕ_{22}	0.00	0.00	0.00	0.61	0.47	0.00	0.02	0.47	0.66
	c	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S2	ϕ_{10}	0.00	0.00	0.00	0.54	0.00	0.00	0.51	0.08	0.30
	ϕ_{11}	0.00	0.00	0.00	0.69	0.00	0.00	0.20	0.04	0.19
	ϕ_{12}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.28	0.02
	ϕ_{20}	0.00	0.00	0.00	0.00	0.13	0.00	0.94	0.34	0.21
	ϕ_{21}	0.00	0.00	0.00	0.62	0.66	0.00	0.20	0.61	0.44
	ϕ_{22}	0.00	0.00	0.00	0.17	0.93	0.17	0.62	0.27	0.69
	c	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

* "bias" stands for the average bias, "mse" denotes a mean squared error.