Testing for normality of weakly dependent data

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Introduction

The validity of normality of the marginal law (of economic variables) may be very useful in practice:

- **Econometrics**:
  - econometric modelling/identification;
  - forecasting (testing for Gaussian prediction bands)
  - bootstrap techniques;
  - statistical filters.
- **Macro-Finance**:
  - option pricing;
  - risk management;
  - yield curve modelling (testing for Gaussian affine term structure models).

Some basics about normality tests

There are 3 classes of (**i.i.d. based**) tests in the literature:

- Empirical distribution (characteristic) function tests (e.g. Kolmogorov-Smirnov test, Anderson-Darling test);
- Order statistic based tests (e.g. Shapiro-Wilks test);
- Moment based tests (e.g. Jarque-Bera test).

Bai and Ng (2005) modified a JB test for weakly dependent observations (the only test for w.d.). But the test suffers from some shortcomings

- The finite-sample properties of the test are very poor;
- The test requires the first eight moments to be finite.

The main task of the paper

The main task is to propose a new version of some existing test for normality of weakly dependent data under minimal (moment) conditions.

Which test to modify?

The Anderson-Darling (AD) test is one of the most power test of normality in the literature. The AD test takes the form

$$\mathcal{A}_n = \int_{\mathbb{R}} \frac{(F_n(z) - \Phi(z))^2}{\Phi(z)(1 - \Phi(z))} \, \mathrm{d}\Phi(z), \quad z \in \mathbb{R}, \tag{1}$$

where Φ denotes a standard normal distribution and F_n is the empirical distribution function associated with $\{X_t : t = 1, \ldots, n\}$

$$F_n(z) = \frac{1}{n} \sum_{t=1}^{n} I\left(\frac{X_t - \mu}{\sigma} \leq z\right), \quad z \in \mathbb{R}, \tag{2}$$

where $I(\cdot)$ is a standard indicator function and $\mu = \mathbb{E}(X_t)$ and $\sigma = \sqrt{\mathrm{var}(X_t)}$. It can be shown that as $n \to \infty$ and for given μ and σ, then

$$n\mathcal{A}_n \xrightarrow{d} \int_0^1 \frac{U^2(\omega)}{\omega(1 - \omega)} \, \mathrm{d}\omega, \tag{3}$$

where U is the Brownian bridge.

Complications with the AD test statistic

We face 2 complications when using the AD test in practice:

- Complication 1: once the parameter(s) μ and/or σ in (2) are **unknown** and must be estimated from data. The asymptotic distribution is no longer parameter free ((3) does not hold);
- Complication 2: economic time series (at least some of them) can be characterized as **weakly dependent** process and the asymptotic distribution is no longer parameter free;
- An appropriate bootstrap method has to be used in order to (i) replicate the dependence in data; (ii) impose the normality assumption under H_0.
- We implement an AR-sieve bootstrap to calculate the critical values of the AD test.

Assumptions about the stochastic process

Assumption 1 The underlying stochastic process $\{X_t\}$ is a real-valued stationary and weakly dependent process allowing for a Wold representation given by

$$X_t = \mu + \sum_{j=1}^{\infty} \psi_j \epsilon_{t-j} + \epsilon_t, \quad t \in \mathbb{Z}, \tag{4}$$

where $\mu \in \mathbb{R}$, the roots of the lag polynomial $\psi(q) = 1 - \sum_{j=1}^{\infty} \psi_j q^j$ lie outside the unit disk and $\sum_{j=1}^{\infty} j|\psi_j| < \infty$, the error sequence $\{\epsilon_t\}$ is assumed to be stationary and ergodic such that $\mathbb{E}(\epsilon_t|\mathcal{F}_{t-1}) = 0$, $\mathbb{E}(\epsilon_t^2|\mathcal{F}_{t-1}) = s^2 < \infty$, where $\mathcal{F}_t = \{\epsilon_t, \epsilon_{t-1}, \dots\}$ is the σ-field, $\mathbb{E}(\epsilon_t^4) < \infty$ and the density function $f(\epsilon_t)$ is absolutely continuous.

Note

Under an additional mild assumption the process in (4) can be written into an AR(∞) model.

AR-sieve bootstrap

Algorithm 1

(i) Select an appropriate lag order p of an AR model using the AIC.

(ii) Estimate the unknown AR(p) model parameters by the OLS.

(iii) Construct a sequence of the estimated residuals $\{\hat{\epsilon}_t : t = p+1, \ldots, n\}$ by the recursion

$$\hat{\epsilon}_t = X_t - \hat{c} - \sum_{i=1}^{p} \hat{\phi}_i X_{t-i}.$$

(iv) Under the null hypothesis of marginal normality, the hypothesized distribution equals to a standard normal distribution Φ. Therefore, consistently with the null, draw independent random errors $\epsilon_t^* \sim N(0, \hat{s}^2)$, for $t = 1, \ldots, n+100$, where $\hat{s}^2 = (n-2p-1)^{-1} \sum_{t=p+1}^{n} \hat{\epsilon}_t^2$.

AR-sieve bootstrap

Algorithm 1

(v) Generate bootstrap replicates $\{X_t^* : t = 1, \ldots, n + 100\}$ by the recursion

$$X_t^* = \hat{c} + \sum_{i=1}^{p} \hat{\phi}_i X_{t-i}^* + \epsilon_t^*,$$

where the process is initiated by a vector of sample averages: $(X_{-p+1}^*, \ldots, X_0^*) = (\bar{X}, \ldots, \bar{X})$. The first 100 data points are then discarded in order to eliminate start-up effects and the remaining n data points are used.

(vi) Construct a bootstrap analogy of the BAD test $\mathcal{A}_n^*$ calculated from a bootstrap sample $\{X_t^* : t = 1, \ldots, n\}$.

AR-sieve bootstrap

Algorithm 1

(vii) Repeat steps (iv)–(vi) independently B times to get a sample of the BAD statistics $\{\mathcal{A}^*_{n,i} : i = 1, \dots, B\}$. Then, the sampling distributions of the BAD test statistic is approximated by the empirical distribution functions associated with $\{\mathcal{A}^*_{n,i} : i = 1, \dots, B\}$: $H^*_n(u) = B^{-1} \sum_{i=1}^{B} I(\mathcal{A}^*_{n,i} \leq u)$. Finally, a bootstrap test of the nominal level α rejects the null hypothesis of normality if

$$\hat{\mathcal{A}}_n > \inf\{u : H^*_n(u) \geq (1 - \alpha)\},$$

where $\hat{\mathcal{A}}_n$ is the BAD test statistic obtained from the observed sample $\{X_t : t = 1, \dots, n\}$.

Multivariate extension

- Since the estimation of the multivariate EDF-based tests is computationally expensive, some dimensionality reduction technique is desirable for multiple time series applications;
- A natural solution utilizes the fact that if a $(k \times 1)$ random vector $\boldsymbol{x}_t$ is distributed as $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ under H_0, then $y_t = (\boldsymbol{x}_t - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}_t - \boldsymbol{\mu})$ is distributed as $\chi^2(k)$;
- Requires the VAR-sieve bootstrap = not very convenient in our case;
- We introduce a simple, linear, yet flexible, dimensionality reduction method which is in spirit similar to the quadratic-form one. The method is motivated by the well-known Cramér-Wold theorem.

Multivariate extension...

Theorem (Cramér-Wold)

For a $(k \times 1)$ *random vectors* $\boldsymbol{x}_t = (X_{1t}, \ldots, X_{kt})'$ *and* $\boldsymbol{x} = (X_1, \ldots, X_k)'$*, a necessary and sufficient condition for* $\boldsymbol{x}_t \xrightarrow{d} \boldsymbol{x}$ *with a joint distribution* $F(\mathbf{x})$ *as* $t \to \infty$ *is that* $\boldsymbol{\lambda}'\boldsymbol{x}_t \xrightarrow{d} \boldsymbol{\lambda}'\boldsymbol{x}$ *with a marginal distribution function* $F(\boldsymbol{\lambda}'\boldsymbol{x})$ *for each* $\boldsymbol{\lambda} \in \boldsymbol{R}^k$*.*

An ultimate question is, however, how to determine the aggregation vector $\boldsymbol{\lambda}$.

Multivariate extension...

We propose a two-step procedure (orthogonalization and aggregation):

- Step 1: The orthogonalization can be done by the eigenvalue decomposition which decomposes a $(k \times k)$ symmetric and positive-definite variance-covariance matrix as follows: $\mathrm{cov}(\boldsymbol{x}_t) = \boldsymbol{\Sigma} = \boldsymbol{P}\boldsymbol{P}'$, where $\boldsymbol{P}$ is a square matrix;
- Step 2: Orthogonalized components $\boldsymbol{z}_t = \boldsymbol{P}^{-1}\boldsymbol{x}_t = (Z_{1t}, \ldots, Z_{kt})'$ are then aggregated using the skewness-based weighting function defined as $\boldsymbol{w} = [w_i]$, where $w_i = 1$ if $skew(Z_{it}) \geq 0$ and $w_i = -1$ if $skew(Z_{it}) < 0$, for $i \in \{1, \ldots, k\}$, and $skew(\cdot)$ can by any measure of skewness;
- Finally, one can then apply the testing procedure described in Algorithm 1 to the transformed scalar process $X_t = \boldsymbol{\lambda}'\boldsymbol{x}_t$, where $\boldsymbol{\lambda} = \boldsymbol{w}'\boldsymbol{P}^{-1}$.

Monte Carlo setup

The finite-sample properties of the BN and BAD tests are assessed using the following DGPs:

M1: $X_t = 0.5X_{t-1} + \varepsilon_t$

M2: $X_t = 0.8X_{t-1} + \varepsilon_t$

M3: $X_t = 0.8X_{t-1} - 0.4X_{t-2} - 0.5\varepsilon_{t-1} + \varepsilon_t$

M4: $X_t = 0.5X_{t-1} - 0.3X_{t-1}\epsilon_{t-1} + \epsilon_t$

M5: $X_t = 1.5S_t - 0.5(1 - S_t) + 0.5X_{t-1} + \epsilon_t$

M6:

$$\boldsymbol{x}_t = \begin{pmatrix} 0.4 & 0.3 \\ 0.3 & 0.4 \end{pmatrix} \boldsymbol{x}_{t-1} + \begin{pmatrix} a_t \\ \varepsilon_t \end{pmatrix}$$

where $a \sim N(0,1)$ and $\varepsilon \in \{N(0,1), S1, S2, S3, A1, A2, A3\}$.

Monte Carlo setup...

Table : Parameters of Generalized Lambda Distribution

	λ_1	λ_2	λ_3	λ_4	skewness	kurtosis
S1	0.000000	-1.000000	-0.080000	-0.080000	0.0	6.0
S2	0.000000	-0.397912	-0.160000	-0.160000	0.0	11.6
S3	0.000000	-1.000000	-0.240000	-0.240000	0.0	126.0
A1	0.000000	-1.000000	-0.007500	-0.030000	1.5	7.5
A2	0.000000	-1.000000	-0.100900	-0.180200	2.0	21.1
A3	0.000000	-1.000000	-0.001000	-0.130000	3.2	23.8

Monte Carlo results

		$n = 100$		$n = 200$		$n = 500$	
DGP	distr.	BN	BAD	BN	BAD	BN	BAD
M1	N	0.03	0.05	0.05	0.05	0.09	0.04
	S1	0.01	0.23	0.04	0.40	0.22	0.78
	S2	0.04	0.44	0.09	0.69	0.34	0.97
	S3	0.04	0.63	0.11	0.88	0.33	1.00
	A1	0.21	0.81	0.83	0.97	1.00	1.00
	A2	0.10	0.67	0.35	0.89	0.79	1.00
	A3	0.45	1.00	0.97	1.00	1.00	1.00
M2	N	0.01	0.06	0.02	0.06	0.04	0.05
	S1	0.00	0.11	0.00	0.14	0.02	0.17
	S2	0.01	0.19	0.02	0.24	0.06	0.37
	S3	0.01	0.28	0.03	0.39	0.09	0.65
	A1	0.00	0.25	0.03	0.43	0.46	0.80
	A2	0.01	0.32	0.06	0.44	0.36	0.77
	A3	0.00	0.59	0.03	0.88	0.73	1.00

Why does it work?

- The formal proof is given in Bühlmann (1997), some modifications are, however, necessary,
- **An intuitive explanation:** Bickel and Bühlmann (1997) explain that the closure of the Wold representation is fairly large. It means that for any non-linear stochastic process there exist another process in the closer of linear processes having identical sample paths with probability exceeding $1/e \approx 0.37$;
- **A rule of thumb for applications:** Kreiss et al. (2011) prove that if the distribution of a relevant statistic is determined solely by the first two moments, then the AR-sieve bootstrap is expected to work.

Example: symmetric or asymmetric fan-charts?

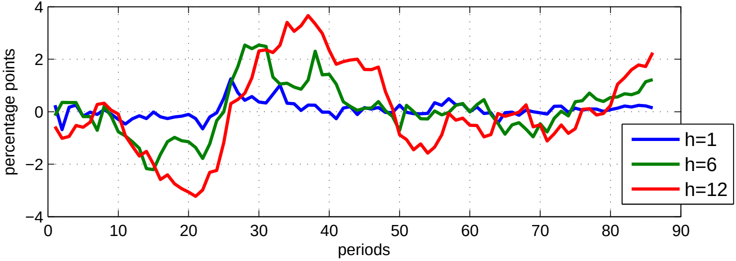
- Prediction bands (fan-charts) have become a standard tool of central banks in assessing risk about the future development of the economy (see Clements and Hendry (2008, Chap. 2,3));
- But we hold the view that **only correct prediction bands** may be useful in practice;
- Researches face an ultimate dilemma whether to construct symmetric or asymmetric fan-charts (**symmetric** = Bank of Canada, Riksbank, Norges Bank; **asymmetric** = Bank of England, NBS, National Bank of Poland, Bank of Italy);
- Other applications of the BAD test: calculating the probability of deflation; density forecast evaluation (DSGE vs. (B)VAR models).

Example: NBS inflation forecast errors

- The forecast error is defined as $X_t(h) = \pi_{t+h} - \pi_t(h)$ for a forecast horizon $h \in \{1, \ldots, 12\}$, where $\pi_t(h)$ stands for the h-step ahead forecast of the (year-on-year) CPI inflation rate and π_{t+h} denotes the actual inflation rate.
- We focus on testing for marginal and joint normality of the NBS inflation forecast errors:
  - marginal hypothesis:
    $H_0 : F(X_t(h)) = N(0, \sigma_h^2)$ against
    $H_1 : F(X_t(h)) \neq N(0, \sigma_h^2)$;
  - joint hypothesis:
    $H_0 : F(X_t(1), \ldots, X_t(12)) = N(\mathbf{0}, \mathbf{\Sigma})$ against
    $H_1 : F(X_t(1), \ldots, X_t(12)) \neq N(\mathbf{0}, \mathbf{\Sigma})$.

Example: symmetric or asymmetric fan-charts?

Figure : NBS Inflation Forecast Errors



Example: symmetric or asymmetric fan-charts?

Table : P-values of the BAD Test for Marginal and Joint Normality of the Inflation Forecast Errors

hypothesis	horizon	$B = 1000$	$B = 5000$	$B = 10000$
marginal	$h = 1$	0.04	0.04	0.04
	$h = 2$	0.36	0.35	0.35
	$h = 3$	0.24	0.24	0.24
	$h = 4$	0.40	0.41	0.40
	$h = 5$	0.49	0.50	0.50
	$h = 6$	0.63	0.61	0.63
	$h = 7$	0.91	0.89	0.87
	$h = 8$	0.84	0.86	0.86
	$h = 9$	0.87	0.86	0.87
	$h = 10$	0.73	0.74	0.73
	$h = 11$	0.57	0.59	0.57
	$h = 12$	0.56	0.59	0.58
joint	$h = 1, \ldots, 12$	0.65	0.62	0.62

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Thanks

Thank you for attention.